Interaction Term of Tsai-Wu Theory for Laminated Veneer

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Abstract: A stochastic-based method of evaluating the interaction parameter (F_{12}) of the Tsai-Wu strength theory has been presented in this paper. Treating all strength parameters of the strength theory as random variables, the mean and standard deviation of F_{12}, under plane stress conditions, have been estimated for Douglas-fir laminated veneer. This estimation has been managed through a nonlinear least-squares fit of the parameters to a cumulative probability distribution of off-axis tensile data. For the purpose of comparison, a sample evaluation of F_{12} using deterministic methods has been presented. This evaluation showed strong dependence of F_{12} on angle to grain. A subsequent sensitivity analysis of the off-axis tests on the value of F_{12} indicated that data from 15° off-axis tensile tests were more stable in establishing F_{12} than that of other angles tested: 30°, 45°, and 60°.

INTRODUCTION

In keeping with reliability-based design methodology, present strength prediction techniques should be based on probability theory, taking into account the natural variability of material properties. Such refinement in the assessment of material strength will inevitably lead to safer, more reliable designs, as discussed by Foschi (1990). As well, a probabilistic strength prediction approach can lead to more efficient use of the material, which is particularly significant to wood or wood-based materials as pressures on the world's timber resources increase. The present paper demonstrates how probability theory may be incorporated in the formulation of the Tsai-Wu strength theory (Tsai and Wu 1971) to be used in predicting the strength of Douglas-fir laminated veneer.

Prediction of laminated veneer strength under uniaxial stress along the material symmetry axes is elementary, as these strengths are determined through simple uniaxial stress tests. Most practical applications involve multiaxial stress states, however, in which normal and shear stresses act simultaneously. In this complex stress state, member capacity can be predicted through the use of a multiaxial strength theory (or failure criterion).

A multitude of orthotropic strength theories have been developed in the past, and many are reviewed by Sandhu (1972), Hashin (1980), Rowlands (1985), and Nahas (1986). All of these criteria are phenomenological; that is, their basic premise is prediction of material failure without direct reference to actual failure mechanisms. From all of the strength criteria available, no one theory is suitable for all materials and loading conditions. However, the tensor polynomial theory, made popular by Tsai and Wu (1971), has received widespread attention due to its simplicity and generality.

TSAI-WU STRENGTH THEORY

The Tsai-Wu strength theory predicts that failure will occur when the following inequality is satisfied:

$$\mathbf{F}_i\sigma_i + \mathbf{F}_{ij}\sigma_i\sigma_j \geq 1 \quad (1)$$

where $i, j = 1, 2, \ldots, 6$ (repeated indices imply summation); and $\mathbf{F}_i$ and $\mathbf{F}_{ij}$ = second and fourth rank tensors, respectively. For the case of plane stress, shown in Fig. 1, stresses associated with the third axis are considered negligible. Further, when the stresses are defined in the principal material directions, known as special orthotropy, the F_6, F_{16}, and F_{26} terms are zero. Under these conditions, (1) becomes

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + 2F_{12}\sigma_1\sigma_2 + F_{66}\sigma_6^2 \geq 1 \quad (2)$$

The coefficients F_1–F_{66}, with the exception of F_{12}, are described in terms of the strengths in the principal material directions. Referring to Fig. 2, the five principal strengths are: tension and compression parallel to the direction of the fiber (X_t and X_c), tension and compression transverse to the direction of the fiber (Y_t and Y_c), and shear in this same plane (S). Considering a uniaxial tension load on a specimen in the 1 direction, (2) at failure becomes

$$F_1X_t + F_{11}X_t^2 = 1 \quad (3a)$$

and for compression, (2) at failure becomes

$$F_1X_c + F_{11}X_c^2 = 1 \quad (3b)$$

By solving (3*a*) and (3*b*) simultaneously and regarding the compression strength as negative, the expressions for the strength parameters F_1 and F_{11} are found to be

$$F_1 = \frac{1}{X_t} - \frac{1}{X_c}; \quad F_{11} = \frac{1}{X_tX_c} \quad (4a)$$

Through similar mathematical manipulations, it can be shown that

$$F_2 = \frac{1}{Y_t} - \frac{1}{Y_c}; \quad F_{22} = \frac{1}{Y_tY_c}; \quad F_{66} = \frac{1}{S^2} \quad (4b)$$

The remaining unknown strength parameter of (2), F_{12}, accounts for the interaction between normal stresses, σ_1 and σ_2. As such, its evaluation must occur under a biaxial loading condition in which both normal stresses are nonzero. Further, the magnitude of F_{12} is constrained by the stability condition

$$F_{11}F_{22} - F_{12}^2 \geq 0 \quad (5)$$

to ensure closure of the failure surface. Violation of this con-

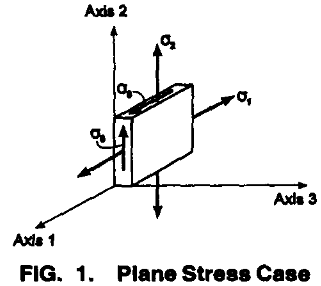

FIG. 1. Plane Stress Case

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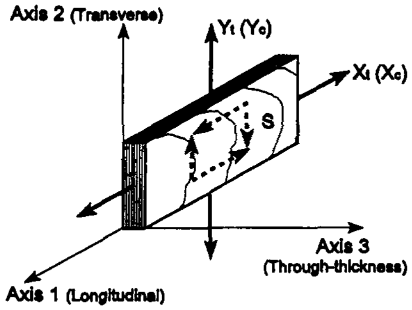


FIG. 2. Principal Strengths and Coordinate System of Laminated Veneer

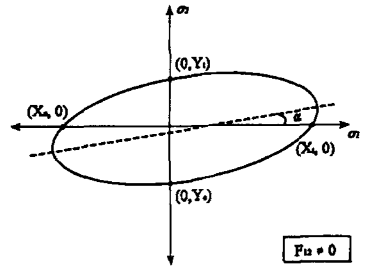


FIG. 3. Sample Tsai-Wu Failure Surface (σ_6 = Constant)

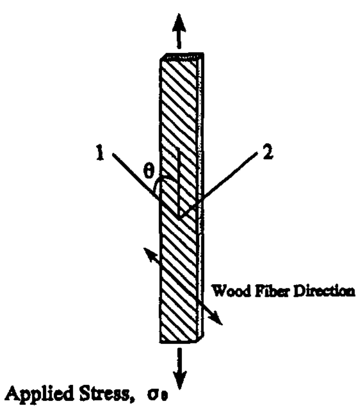


FIG. 4. Off-Axis Tensile Test

dition would imply infinite strength for some stress states, which is physically impossible, in plane stress.

The surface forms an ellipsoid in stress space, as shown in Fig. 3, where F_{12} characterizes the rotation (α) of the ellipsoid with respect to the stress-coordinate axes. In this regard, the parameter is an important component of the strength theory. In fact, it has been argued that the value of F_{12} "typically determines the effectiveness of tensorial-type failure criteria" (Suhling et al. 1984).

In past studies, difficulties have been encountered when evaluating F_{12} experimentally due to its sensitivity to experimental variation. Slight inaccuracies in measurements of strength resulted in large inaccuracies in the calculated value of F_{12}, which was aggravated by the fact that in order for F_{12} to be physically admissible, it must satisfy the prescribed stability bounds [(5)]. Even after careful experimental investigation, several studies reported unacceptable results for F_{12}. For example, Pipes and Cole (1973) found that from a series of off-axis tests (Fig. 4) on boron-epoxy specimens with angles to grain (θ) of 15°, 30°, 45°, and 60°, only the 15° data produced values for F_{12} that satisfied the stability criterion. This result is particularly interesting in that the same result was obtained for laminated veneer in the study presented here, as will be shown subsequently. Suhling et al. (1984) also found that due to the highly sensitive and unstable nature of F_{12}, the off-axis test method was not suitable in determining the interaction parameter for paperboard.

These difficulties prompted other researchers to seek theoretical solutions to the problem. For example, Narayanaswami and Adelman (1977) asserted that the arbitrary assignment of F_{12} equal to zero was acceptable for filamentary composites. Also, Cowin (1979), van der Put (1982), and Liu (1984) suggested various formulas to calculate F_{12} based on the well-known Hankinson formula. Despite these efforts, a standard method of determining F_{12} was never established. In light of this, the present paper investigates the determination of F_{12} using probability theory. This statistical approach is a more rational approach for dealing with variable properties and should ultimately provide a more reliable strength theory. For the purpose of comparison, a deterministic evaluation of F_{12} is first presented. Both the deterministic and statistical evaluation were conducted using Douglas-fir laminated veneer.

Experimental Data

Experiments were performed to procure the principal material strengths of Douglas-fir laminated veneer. Also, a series of off-axis tensile tests for grain angles of 15°, 30°, 45°, and 60° were performed. Specimens were cut from 19 individual boards comprised of 11 3.2 mm × 1,220 mm × 2,440 mm laminated sheets. Fabrication specifications, material treatment, and test methods conforming to appropriate ASTM standards are reported in Clouston (1995). All specimens were prepared in the same manner to control strength variations due to environmental conditions. Also, all tensile specimens were of equal size (610 mm long by 63 mm wide by 35 mm thick) to control size effect. Shear strength, however, was evaluated on standard ASTM shear block specimens, and therefore required a shear size adjustment factor. This factor was determined using Weibull weakest-link theory (Weibull 1939), described in detail in Clouston (1995). A comprehensive summary of all measured strength data is given in Table 1. Descriptive statistics are provided for each set of data.

DETERMINISTIC EVALUATION OF F_{12}

Analytical

Using the mean strength values shown in Table 1, the mean values for $\mathbf{F}_i$ and $\mathbf{F}_{ij}$ are calculated from (4*a*) and (4*b*) as

$$\mathbf{F}_i = \begin{Bmatrix} F_1 \\ F_2 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} +6.25 \times 10^{-4} \\ +3.61 \times 10^{-1} \\ 0 \end{Bmatrix} \text{MPa}^{-2} \qquad (6a)$$

TABLE 1. Summary of Data

Material strength (or property) (1)	Count (2)	Mean (MPa) (3)	Standard deviation (MPa) (4)	Coefficient of variation (%) (5)
0° (X_t)[a]	18	55.31	10.11	18.28
15°[a]	17	18.92	1.39	7.35
30°[a]	17	6.47	0.40	6.18
45°[a]	18	3.74	0.24	6.42
60°[a]	16	2.68	0.18	6.72
90° (Y_t)[a]	17	2.25	0.22	9.78
Compression parallel (X_c)	18	57.29	2.93	5.11
Compression perpendicular (Y_c)	18	12.02	1.38	11.48
Shear (S)	19	11.02	1.17	10.62
Shear size adjustment factor	—	0.72	0.063	8.75
Moisture content (%)	103	7.91	0.26	3.29
Specific gravity (g/cm^3)	103	0.53	0.02	3.77

[a]Tension by grain angle.

$$\mathbf{F}_{ij} = \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ & F_{22} & F_{26} \\ \text{sym} & & F_{66} \end{bmatrix}$$

$$= \begin{bmatrix} +3.16 \times 10^{-4} & F_{12} & 0 \\ & +3.70 \times 10^{-2} & 0 \\ \text{sym} & & +1.59 \times 10^{-2} \end{bmatrix} \text{MPa}^{-2} \quad (6b)$$

It should be noted that the shear strength used in calculating F_{66} was first adjusted for size effect. This was done by multiplying the mean experimental shear strength by the mean shear size adjustment factor such that the value $F_{66} = 1/(S \cdot \text{adjustment factor})^2$.

As the off-axis test produces a complex stress state with respect to the material's principal axes, its results can be used to calculate a mean strength value for the interaction parameter, F_{12}. Referring to Fig. 4, the applied stress, σ_θ produces the following stresses along the principal material directions:

$$\sigma_1 = \sigma_\theta \cos^2\theta; \quad \sigma_2 = \sigma_\theta \sin^2\theta; \quad \sigma_6 = -\sigma_\theta \cos\theta \sin\theta \quad (7)$$

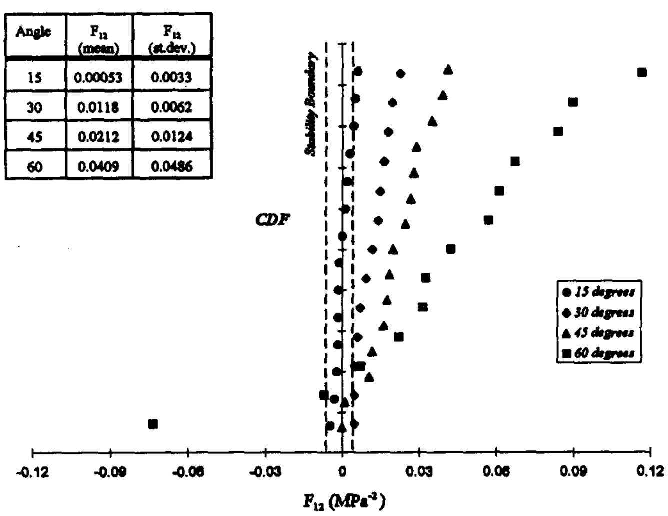
Substituting (7) into (2) and rearranging yields a solution for F_{12}

$$F_{12} = \frac{1}{2\sigma_\theta^2}\left[\left(\frac{1}{\sin^2\theta \cos^2\theta}\right) - \left(\frac{F_1}{\sin^2\theta} + \frac{F_2}{\cos^2\theta}\right)\sigma_\theta - \left(F_{66} + \frac{F_{11}}{\tan^2\theta} + F_{22}\tan^2\theta\right)\sigma_\theta^2\right] \quad (8)$$

From (8), a different mean value of F_{12} for each angle to grain can be calculated. For example, considering the 15° off-axis strength data, $F_{12} = +0.00039$ MPa^{-2}; however, for the 60° off-axis strength data, $F_{12} = +0.038$ MPa^{-2}. Furthermore, only the 15° data satisfy the stability criterion for which the deterministic upper and lower bounds are ± 0.0034 MPa^{-2}. Thus, using this deterministic approach with off-axis data produces inconsistent and unreliable results. To gain further insight into these results, a probabilistic evaluation of F_{12} for each angle to grain was conducted.

Using (8), F_{12} was computed for the four angles to grain for each of the 19 boards tested. The results are displayed in Fig. 5 as cumulative probability distributions of F_{12} for each angle to grain.

Each probability distribution is quite distinct. The F_{12} value associated with the 15° data is fairly consistent (standard deviation = 0.0033 MPa^{-2}) with a mean value of -0.00053 MPa^{-2}. In contrast, the values from the larger angles are less consistent with larger mean values. This phenomenon can be explained partially by a high sensitivity of F_{12} to variations in experimental data.



Angle	F_{12} (mean)	F_{12} (st.dev.)
15	0.00053	0.0033
30	0.0118	0.0062
45	0.0212	0.0124
60	0.0409	0.0486

FIG. 5. Cumulative Distribution Function of F_{12} for Each Angle to Grain

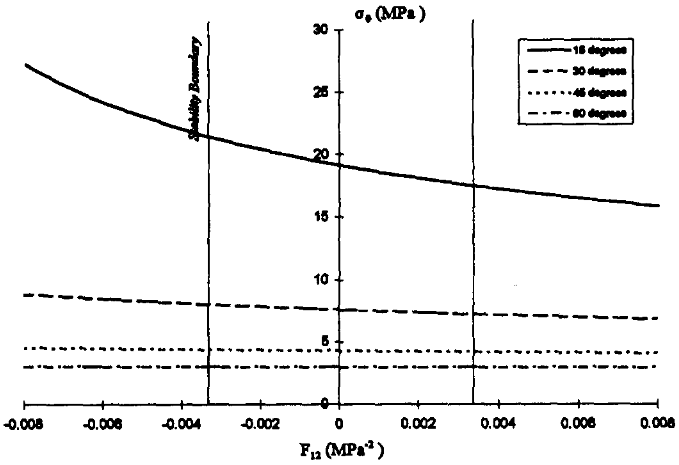


FIG. 6. Sensitivity of F_{12} for Each Angle to Grain

Sensitivity of F_{12} to Off-Axis Experimental Data

Following an interpretation technique similar to that proposed by Tsai and Wu (1971) to illustrate the effect of variations in data on F_{12}, a stress versus F_{12} plot has been created in Fig. 6, displaying a different curve for each angle to grain. This plot illustrates that curves for 30°, 45°, and 60° are nearly horizontal. This means that if a small inaccuracy is made in measuring the 30°, 45°, or 60° off-axis strengths (from human or systematic error), the resulting calculated value for F_{12} would vary extensively and would be completely obscured in the stability region. This is likely the reason for the high variability of F_{12} for these three angles, shown in Fig. 5. The 15° curve is slightly more inclined, however, meaning that minor inaccuracies in the experimental data will not greatly affect the calculated value of F_{12}.

Accepting the fact that experimental results have inaccuracies, then a more accurate estimation of the "true" value of F_{12} can be had by considering the 15° data only. Given this "true" value, the reverse approach, calculating strength given F_{12}, would be valid for any angle. This would not necessarily be the case if all four angles were used. Thus, the 15° off-axis test is the most reliable of the four investigated, and the following statistical analysis to determine F_{12} incorporates experimental results from this series only.

STATISTICAL EVALUATION OF F_{12}

A nonlinear, least-squares minimization procedure was executed to approximate the mean and standard deviation of F_{12}. For this procedure, the program "DOLFIT," for fitting parameters to the Foschi-Yao damage accumulation model (Foschi and Yao 1986), was adapted. In short, the function

$$\Phi = \sum_{i}^{N\text{prob}} \left(1 - \frac{\sigma_{\theta_i}^{\text{pred}}}{\sigma_{\theta_i}^{\text{exp}}}\right)^2 \quad (9)$$

was minimized with respect to the mean and standard deviation of F_{12}, where "N prob" denotes number of probability levels for consideration; and superscripts "pred" and "exp" refer to "predicted" and "experimental" off-axis strengths, respectively. Calculation of the predicted off-axis strength was formulated as follows. Substituting (7) into (2) yields

$$F_1\sigma_\theta \cos^2\theta + F_2\sigma_\theta \sin^2\theta + F_{11}\sigma_\theta^2 \cos^4\theta + F_{22}\sigma_\theta^2 \sin^4\theta + 2F_{12}\sigma_\theta^2 \cos^2\theta \sin^2\theta + F_{66}\sigma_\theta^2 \sin^2\theta \cos^2\theta = 1 \quad (10)$$

Rearranging, we get

$$\sigma_\theta^2(F_{11}\cos^4\theta + F_{22}\sin^4\theta + 2F_{12}\cos^2\theta\sin^2\theta + F_{66}\cos^2\theta\sin^2\theta) + \sigma_\theta(F_1\cos^2\theta + F_2\sin^2\theta) - 1 = 0 \quad (11)$$

Now designating

$$X_1 = F_1\cos^2\theta + F_2\sin^2\theta;\quad X_2 = F_{11}\cos^4\theta + F_{22}\sin^4\theta + F_{66}\cos^2\theta\sin^2\theta;\quad X_3 = 2\cos^2\theta\sin^2\theta \quad (12)$$

σ_θ can be expressed as

$$\sigma_\theta = \frac{-X_1 \pm [X_1^2 + 4(X_2 + F_{12}X_3)]^{1/2}}{2(X_2 + F_{12}X_3)} \quad (13)$$

Thus, utilizing (13), the off-axis strength can be modeled as a function of the principal strengths, grain angle, and the unknown parameter, F_{12}. The following details of the minimization process are presented for completeness.

Minimization Procedure

To begin, arrays consisting of 2,500 values for each principal strength (X_t, X_c, Y_t, Y_c, S), and F_{12} were randomly generated. In doing so, lognormal distributions were chosen to represent each principal strength, as they were found to describe these strengths relatively well by Clouston (1995). A normal distribution was deemed appropriate for F_{12}, as it enabled either positive or negative values, reflecting this characteristic of F_{12}. Statistical data used for generation of the principal strengths are shown in Table 1, whereas initial values for the mean and standard deviation of F_{12} were estimated by the user. Values of each of the principal strengths and F_{12} were generated independently of each other since they showed no significant correlation from the basic unidirectional test data.

For each set of randomly generated values, the predicted off-axis strength (σ_θ^{pred}) was calculated using (13). This predicted strength was ranked (i.e., sorted in ascending order and given an appropriate probability of failure). The residual function, Φ [(9)], was then calculated, where the predicted strength (σ_θ^{pred}) was determined for the same probability levels (i) as the experimental strength (σ_θ^{exp}).

Based on the gradient of the residual function (with respect to mean and standard deviation of F_{12}), adjusted values of the mean and standard deviation of F_{12} were computed. The gradient was estimated by a perturbation process. These new (adjusted) statistical parameters replaced the initial estimated values, and the residual function was reevaluated.

This procedure was repeated until the difference between residual function values for subsequent iterations satisfied a set tolerance, ensuring convergence for the final solution.

As this procedure was based on a minimization technique, there was potential for the solution to be found at a local minimum rather than a global minimum; therefore, it was sensitive to the initial input values. For this reason, several initial values were checked and the solution yielding the smallest function Φ was deemed to be the final solution.

RESULTS

This analysis produced a mean value of $F_{12} = +0.00003$ MPa^{-2} and a standard deviation of 0.000015 MPa^{-2}. As no data exist from other studies with which to compare this result, we have assessed it based on its conformity to the stability bounds. As previously stated, using the deterministic average values of the strength parameters in (5), the bounds were found to be ± 0.0034 MPa^{-2}. Obviously, the mean value falls well within the deterministic bounds.

For visual interpretation of these results, the off-axis strengths were randomly generated according to (13) with 400 replications. The results were ranked and plotted as cumulative distribution functions in Fig. 7. The Tsai-Wu model obviously fits the experimental 15° data very well, as would be expected since F_{12} was fitted to these data. The 30°, 45°, and 60° data are reasonably accurate; however, they are consistently overestimated.

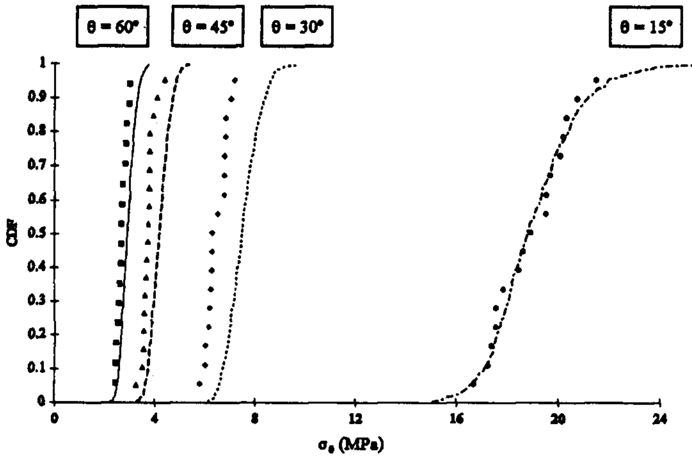


FIG. 7. Tsai-Wu Random Variable Model versus Experimental Off-Axis Tensile Results

It is speculated that this liberal estimate of the higher angles is a result of the testing apparatus. The off-axis tests were conducted with the use of nonrotating clamped grips. Pagano and Halpin (1968) showed that these end constraints could induce shearing forces and bending couples at the ends of the specimens. Further to this, however, Rizzo (1969) showed how these nonuniform influences could be minimized by providing an adequate length (l) to width (w) ratio. He found that for long specimens with $l/w \simeq 10$, "a high degree of test accuracy (could) be obtained." The specimens in the present study had a $l/w = 9.7$, which should be adequate to provide sufficiently accurate results; however, a small grip effect may have produced higher stresses than predicted with classical mechanics [(7)], resulting in lower observed off-axis strengths.

CONCLUSIONS

The present paper has provided insight into the evaluation of the interaction parameter, F_{12}, of the Tsai-Wu theory when measured by way of off-axis tensile tests. Using experimental data for Douglas-fir laminated veneer, it was found that the calculated value of F_{12} was strongly dependent on the off-axis angle to grain. It was further shown, through a sensitivity study, that data from a 15° off-axis test were more reliable than data of other angles tested, 30°, 45°, and 60°, since small inaccuracies in the 15° data would have less impact on the calculated value of F_{12} than those of the larger angles considered. Using the 15° data, a probability-based method of evaluation of F_{12} was demonstrated. This method entailed a nonlinear least-squares minimization process to determine the mean and standard deviation of F_{12}. The results were well within prescribed, deterministic stability bounds, supporting the method proposed herein.

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