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Incorporating size effects in the Tsai-Wu strength theory for Douglas-fir laminated veneer

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Summary This paper elucidates the need to consider the effect of volume on brittle material strengths when these strengths are used in a strength theory. Specifically, Weibull weakest-link theory has been implemented with the Tsai-Wu strength theory to predict the ultimate load carrying capacity of a center point off-axis bending member made from Douglas-fir laminated veneer. Weibull theory has been used in two distinct ways to account for size effects needed to evaluate brittle material strengths (ie. tension perpendicular and parallel to grain, and shear) for the strength criterion. The analytical methods assume linear elastic, plane stress states and have been described and evaluated using probability theory as a framework. Analytical results are in reasonable agreement with experimental findings substantiating the techniques proposed herein.

Introduction

It is widely recognized that the strength of wood and wood-based materials in the brittle failure modes of tension or shear, depend on the volume of stressed material and the nature of the stress distribution (Barrett et al., 1975; Madsen and Buchanan, 1986; Sharp and Suddarth, 1991). These effects have been attributed to the random distribution of strength controlling defects present in the material. It is argued that larger members have a higher probability of containing a larger flaw (or weaker zone) than smaller members and therefore exhibit lower strengths when both volumes are subjected to the same environmental and loading conditions. This has come to be known as 'size effect'.

An important implication of size effect emerges when using strength theories to assess load carrying capacity of a structure or a member. The uniaxial strengths which are incorporated into a multi-axial strength theory must be representative of the volume of material to which the strength theory is being applied. This

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means first adjusting the brittle strengths from representing that of a test volume to representing that of a corresponding volume for the strength theory application. This required adjustment is demonstrated in this paper through application of both Weibull weakest-link theory, to account for size effects, and the Tsai-Wu (tensor polynomial) strength theory (Tsai and Wu, 1971), for prediction of load carrying capacity of an off-axis center point bending member. The Tsai-Wu theory, for orthotropic materials, was chosen for its relative simplicity and generality.

Tsai-Wu theory

The Tsai-Wu theory assumes the existence of a failure surface of quadratic polynomial form. In tensor notation, failure is predicted to occur when

$$F_i\sigma_i + F_{ij}\sigma_i\sigma_j \geq 1 \tag{1}$$

where i, j = 1, 2, ..., 6, and F_i and F_{ij} are 2[nd] and 4[th] rank strength tensors respectively. For the case of plane stress, shown in Fig. 1, stresses associated with the through-thickness direction are neglected and F_6, F_{16} and F_{26} terms are equal to zero due to special orthotropy. Hence, Eq. (1) becomes

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + 2F_{12}\sigma_1\sigma_2 + F_{66}\sigma_6^2 \geq 1 \tag{2}$$

The coefficients F_1 through F_{66}, with exception of F_{12}, are described in terms of the strengths in the principal material directions. Referencing Fig. 1, these strengths are defined as tension and compression parallel to grain (X_t and X_c), tension and compression perpendicular to grain (Y_t and Y_c), and shear in this same plane (S). Through some simple mathematical manipulation, the coefficients of the strength theory are found to be

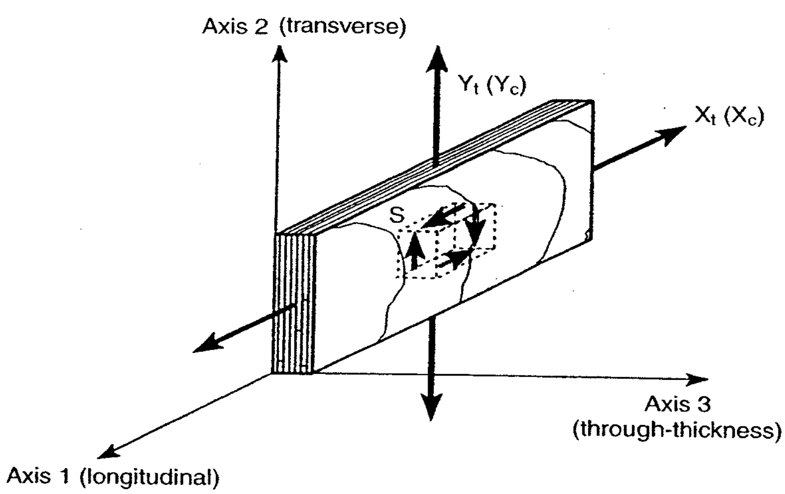

Fig. 1. Coordinate system and principal strengths assuming a plane stress state

$$F_1 = \frac{1}{X_t} - \frac{1}{X_c}$$
$$F_2 = \frac{1}{Y_t} - \frac{1}{Y_c}$$
$$F_{11} = \frac{1}{X_t X_c} \qquad (3)$$
$$F_{22} = \frac{1}{Y_t Y_c}$$
$$F_{66} = \frac{1}{S^2}$$

The remaining unknown parameter of Eq. (2), F_{12}, must be evaluated under a biaxial loading condition to account for the interaction between normal stresses, σ_1 and σ_2. Clouston et al. (1996) describe in detail a stochastic based method to evaluate the statistical parameters (ie. mean and standard deviation) of F_{12} using the same material and material preparation as the current study. (A brief description of this method is also provided herein.) The results of this previous paper, which include statistical data for a complete probability-based Tsai-Wu strength theory for Douglas-fir laminated veneer, were utilized in this current paper to establish the strength tensors in the Tsai-Wu theory.

In order to apply the strength theory with some degree of certainty, all strengths used in the foregoing equations should be derived and subsequently applied under the same conditions. For wood, variables to be controlled include moisture content, temperature, loading time or material volume. In this study, all variables were considered constant with the exception of material volume. That is, the volume of material from which a strength was experimentally determined was not necessarily the same as the volume to which the strength theory was applied. The resulting size effect was managed by way of Weibull weakest-link theory which has been used successfully in previous studies to quantify size effects (Barrett et al., 1975; Lam and Varaglu, 1990; Barrett et al., 1995).

Weibull weakest-link theory

Weibull (1939) used the weakest-link concept to predict the probability of failure of a 'perfectly brittle', homogeneous, isotropic material at a given volume according to the empirically based equation

$$F(\tau) = 1 - e^{-\frac{1}{V_0}\int_v \left[\frac{\tau - \tau_{min}}{m}\right]^k dV} \qquad (4)$$

where:

$F(\tau)$ = probability of failure
τ = material strength
τ_{min} = location parameter (minimum material strength)
m = scale parameter
k = shape parameter
(τ_{min}, m, and k are material constants and V_0 is a reference volume)

Commonly, τ_{min} is accepted as being equal to zero, simplifying the problem, and the resulting formulation is called a two-parameter (2-P) Weibull distribution.

Using 2-P Weibull theory as a basis, a relationship can be formed between the strength for one volume of material and that of another volume where both are at the same probability of failure and subjected to the same stress distribution. The relationship is given as

$$\int_{V_1} \tau_1{}^k \, dV_1 = \int_{V_2} \tau_2{}^k \, dV_2 \tag{5}$$

where the strengths τ_1 and τ_2 correspond to the volumes V_1 and V_2, respectively. The significance of this formulation for this study is that having experimentally determined a strength distribution for a specific test volume, be it for tension or shear, a corresponding distribution can be evaluated for any volume to be used in the strength theory.

Numerical predictions

Experimental strength data were obtained for Douglas-fir laminated veneer and are summarized in Table 1. The material treatment and test methods, which conformed to appropriate ASTM standards for these principal strengths, are reported in Clouston (1995). It is noted that the parallel to grain tensile strength has higher than normal variability when compared to that of commercial LVL (coefficient of variation = 18.3% vs. 9–12% for commercial LVL). This may be explained by the fact that the material in this study did not contain butt joints, unlike conventional LVL, and as a result, the parallel to grain tensile failure behaved more like clear wood than LVL hence showing a larger variability.

The tensile strengths were taken from rectangular specimens, 63 mm × 35 mm × 610 mm in dimensions, under uniform stress conditions. The shear strength was evaluated using a standard ASTM shear block tester. However, the shear block test does not produce a uniform stress field, which makes dealing with Eq. (5) difficult. Thus, shear strength under a uniform stress state needed to be estimated by adjusting the shear block strength by an appropriate factor(α). A nonlinear least square fitting process was used to estimate α and other unknown parameters. The residual function to be minimized for the fitting process can be written as

$$\Phi = \sum_{i}^{N} \left(1 - \frac{\sigma_{\theta_i}^{pred}}{\sigma_{\theta_i}^{exp}} \right)^2 \tag{6}$$

Table 1. Summary of Data

Statistics:	Count	Mean (MPa)	St. Dev. (MPa)	2-P Weibull shape parameter, k
Tension Para., X_t	18	55.31	10.11	5.79
Tension Perp., Y_t	17	2.25	0.22	15.89
Comp. Para., X_c	18	57.29	2.93	–
Comp. Perp., Y_c	18	12.02	1.38	–
Shear, S	19	11.02	1.17	10.61
Shear Adjustment Factor, α	–	0.72	0.63	–
F_{12} (MPa^{-2})	–	+0.00003	0.000015	–
15° off-axis tensile strength, σ_θ(MPa)	17	18.92	1.39	–

where N denotes the number of probability levels for consideration and σ_θ^{pred} and σ_θ^{exp} refer to 'predicted' and 'experimental' off-axis strengths respectively. The off-axis tensile test used to procure experimental off-axis strengths is illustrated in Figure 2. The predicted off-axis strength was calculated as:

$$\sigma_\theta^{pred} = \frac{-X_1 \pm [X_1^2 + 4(X_2 + F_{12}X_3)]^{\frac{1}{2}}}{2(X_2 + F_{12}X_3)} \tag{7}$$

where

σ_θ^{pred} = off-axis tensile strength

$X_1 = F_1 \cos^2\theta + F_2 \sin^2\theta$

$X_2 = F_{11}\cos^4\theta + F_{22}\sin^4\theta + F_{66}\cos^2\theta\ \sin^2\theta$

$X_3 = 2\cos^2\theta\ \sin^2\theta(8)$

$F_{66} = \frac{1}{(S \cdot \alpha)^2}$

α = shear strength adjustment factor

All variables of Eq. (7) (and therefore Eq. (6)) are known with the exception of F_{12} and α. Thus, Eq. (6) was minimized with respect to these two quantities to obtain a least square solution for these variables. The results are presented in Table 1.

Having all necessary strength data, the Tsai-Wu theory was then employed to predict the cumulative probability distribution for load capacity of a center-point off-axis bending specimen (Fig. 3). In an off-axis bending test, the longitudinal axis of the material, axis 1, is oriented at an angle(θ) to the longitudinal axis of the beam. Angles of both 30° and 45° were considered.

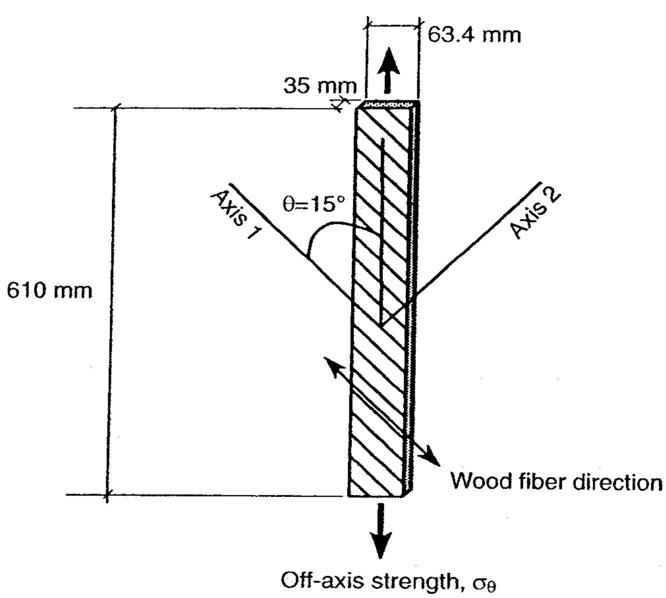

Fig. 2. Off-axis tensile test

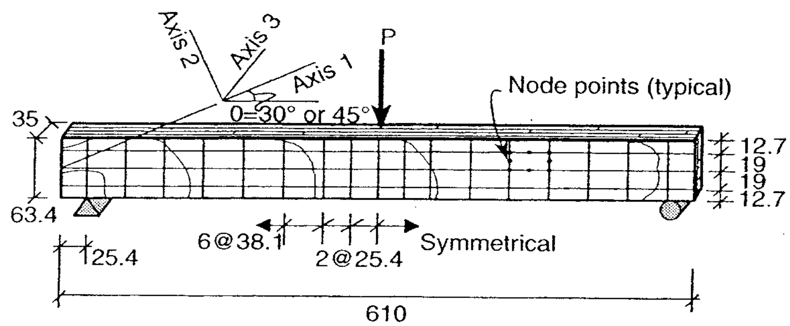

Fig. 3. Off-axis bending specimen and corresponding finite element mesh (units in mm)

The load, in this configuration, induced a predominantly linear elastic failure due to combined tension perpendicular to grain and shear. This scheme was chosen for consistency with the linear elastic off-axis tension test method used to evaluate the interaction parameter, F_{12}.

A linear elastic finite element analysis was performed to estimate the stresses to be used in the prediction model. The orthotropic, plane stress, finite element analysis program 'FEM' written by Foschi, (1974) was used. The mesh consisted of 72 quadratic elements, discretized as shown in Fig. (3). A 3×3 Gauss integration rule was used and the resulting stresses were transformed to the stresses in the principal material directions for direct use in the Tsai-Wu criterion.

The analytical techniques used in this paper are based on Monte Carlo simulations to account for the random nature of the strength parameters. In general, this simulation technique involves substantial repetition of a simulated solution using randomly generated values from assumed probability distributions. In this case, the principal strengths, X_t, X_c, Y_t, etc., were randomly generated according to lognormal distributions, using the experimentally obtained statistical parameters given in Table 1. Lognormal distributions were chosen as they were shown to describe these principal strengths relatively well by Clouston (1995). Based on the randomly generated principal strengths, the strength parameters of the strength theory, F_1, F_2, etc., were computed. In contrast, simulated values for the interaction parameter, F_{12}, were generated directly using the analytically determined mean and standard deviation assuming a normal distribution. A normal distribution was deemed appropriate for F_{12} as it enabled either positive or negative values reflecting this characteristic of F_{12}.

At this stage, all variables of the Tsai-Wu theory (ie. stresses and strength parameters) were available. However, prior to applying the theory, the influence of size on the brittle principal strengths, X_t, Y_t, and S, required consideration. Two analytical methods to combine size effect with the Tsai-Wu theory were investigated.

Method 1 – Direct adjustment of brittle strengths

In this approach, individual beam failure was assessed through evaluation of Eq. (2) over small regions of the beam (typically regions surrounding Gauss integration points within each finite element) which were assumed to be subjected to uniform stresses. For each evaluation, a set of random strength values were generated and considered in conjunction with the applied stresses, σ_1, σ_2, and σ_6 at that point in question. The brittle strengths X_t, Y_t, and S were initially generated using the mean and standard deviation of the strength values based on the original test volume. They were then adjusted for size effect in the following manner.

Given Eq. (5), each one of the brittle strengths may be transformed from representing the strength of the original test volume ($610 \times 63 \times 35\,mm^3$), to representing that of a small volume in the beam (for example, $7.8 \times 35 \times 3.9\,mm^3$, which represents the volume surrounding one Gauss integration point). Since it was assumed that the stresses in both the test volume and the small beam volume were uniform throughout the cross section, Eq. (5) can be simplified to

$$\tau^k_{(test)} V_{(test)} = \tau^k_{(beam)} V_{(beam)} \qquad (9)$$

Rearranging, we can establish a size factor

$$\text{size factor} = \frac{\tau_{(beam)}}{\tau_{(test)}} = \left(\frac{V_{(test)}}{V_{(beam)}}\right)^{\frac{1}{k}} \qquad (10)$$

where k is the appropriate shape parameter for the strength in question taken from Table 1. Thus, the adjusted strength corresponding to a small volume of the beam was found by simply multiplying the known test strength by the size factor.

The brittle strengths were assumed to vary throughout each beam; however, the more ductile strengths were considered to be random between beams but constant within each member.

It was assumed that any indication of material failure meant total collapse of the beam in keeping with brittle fracture theory. The complete beam evaluation was performed 2500 times. Hence, the probability of any beam failing under a specific load was the ratio of the number of beams that failed to the total number of replications, 2500. The entire analysis was carried out for various load levels, enabling the formation of a cumulative probability distribution.

Method 2 – Adjustment of predicted off-axis tensile strengths

A less direct and perhaps less versatile approach to predicting the probability of failure of the bending specimens was investigated to provide a comparison to the former method. This approach exploits the proposition that brittle strength depends on the proportion of material that is highly stressed. According to brittle fracture theory, a pure tension member, which has its entire volume highly stressed, has a higher probability of a critical flaw occurring there and hence a lower strength than does a bending member of the same size which has less volume highly stressed.

In this second method, the Tsai-Wu failure criterion was first used to simulate random strengths of off-axis *tensile* member, oriented at 30° and 45° to grain. These preliminary results are representative of the original brittle strength test volumes only and were calculated per Eq. (7). Then, using Weibull formulation, the off-axis tensile strengths were adjusted to yield the failure stress in the extreme fibre of the off-axis center-point bending specimens at the corresponding angle to grain. The relationship between tensile and bending failure stress is:

$$\tau_{max(b)} = \left(\frac{1}{\left(\frac{V_b}{V_t 2(k+1)^2} \right)^{\frac{1}{k}}} \right) \times \sigma_\theta^{pred} \quad (11)$$

where:
$\tau_{max(b)}$ = maximum tensile stress in a center-point bending specimen (center span at the extreme fibre)
σ_θ^{pred} = off-axis tensile strength

V_b = volume of stressed material in bending specimen
V_t = volume of stressed material in tension specimen
k = 2-P Weibull shape parameter from simulation results
The failure loads in bending were then estimated using elementary beam theory as

$$P = \frac{2bd^2}{3L} \tau_{max\,(b)} \quad (12)$$

where P is the failure load in bending and b, d and L refer to the width, depth and span of the specimen, respectively. The failure load was then ranked to produce a cumulative probability distribution.

Experimental off-axis bending tests

Off-axis bending tests were conducted to provide data for evaluation of the foregoing analytical methods. Descriptive statistics for both the 30° and 45° data are summarized in Table 2. Also, fitted 2-P Weibull coefficients using a maximum likelihood approach are provided as they are used for later analysis. The specimens were conditioned to approximately 8% moisture content and tested under ambient temperature as were those for the uniaxial strength data and the F_{12} parameter. They were tested in the same configuration as that of the analytical model. The predominant failure mode was tension perpendicular to grain with the failure plane coincident with the grain angle. Failure occurred within 51 mm of midspan with one exception in the 45° data set which initiated failure at approximately 100 mm from midspan. The behaviour was consistent with that shown in the prediction model.

Table 2. Off-Axis Bending Test Results

Statistics		Off-Axis Bending Failure Load	
		30°	45°
Count		17	18
Mean (kN)		2.09	1.27
Stand. Dev. (kN)		0.24	0.16
2-P Weibull	Shape, k	10.05	9.01
	Scale, m	2.19	1.34

Results and discussion

The predicted and experimental results of the off-axis bending tests can be visually compared in Figs. (4) through (7). In each case, a 2-P Weibull distribution was fitted to the experimental data to aid in evaluating the predicted results. Figs. (4) and (5) were developed using the first method of incorporating size effects.

The model predictions in Figs. (4) and (5) differ solely in the choice of Weibull shape parameter values for size effect adjustments. The analytical model of Fig. (4) utilized the shape parameters given in Table 1, computed through a maximum likelihood approach. In the model predictions of Fig. (5), the Weibull shape parameters were approximated by $k = (\text{Coefficient of Variation})^{-1.085}$.

Comparing Figs. (4) and (5), it is apparent that the prediction model is dependent on the choice of shape parameters. This is especially important for the shape parameters associated with the perpendicular to grain tensile strength, k_{Yt}. The maximum likelihood and coefficient of variation approach yield $k_{Yt} = 15.89$ and $k_{Yt} = 12.46$, respectively. The smaller 'k' of Fig. (5) resulted in higher tension perpendicular to grain strengths, Y_t and thus produced higher failure loads for the same probability of failure, shifting both the 30° and 45° curves to the right. It resulted in a marginally better overall fit as shown by comparing the sum of squared error (SSE) between results given in Figs. (4) and (5) for the 5^{th}, 50^{th} and 95^{th} percentiles for both 30° and 45° data: SSE = 0.22 and SSE = 0.21.

Although the numerical predictions agree reasonably well with the experimental findings, both the 30° and 45° prediction curves underestimate the variability of the experimental results. This can be attributed to the relatively small sample size of the principal strength data – predominantly the Y_t data, as this is the controlling strength in this load configuration. With a more intense testing scheme, the principal strength variability would be better estimated and hence provide a better overall prediction model.

Fig. (6) illustrates the effectiveness of the second method of incorporating size effect. The results are superior to those of the previous method to which the quantitative evaluation of SSE = 0.11 attests. This improvement may be attributed to a more general dependency on the variability of the principal strengths of

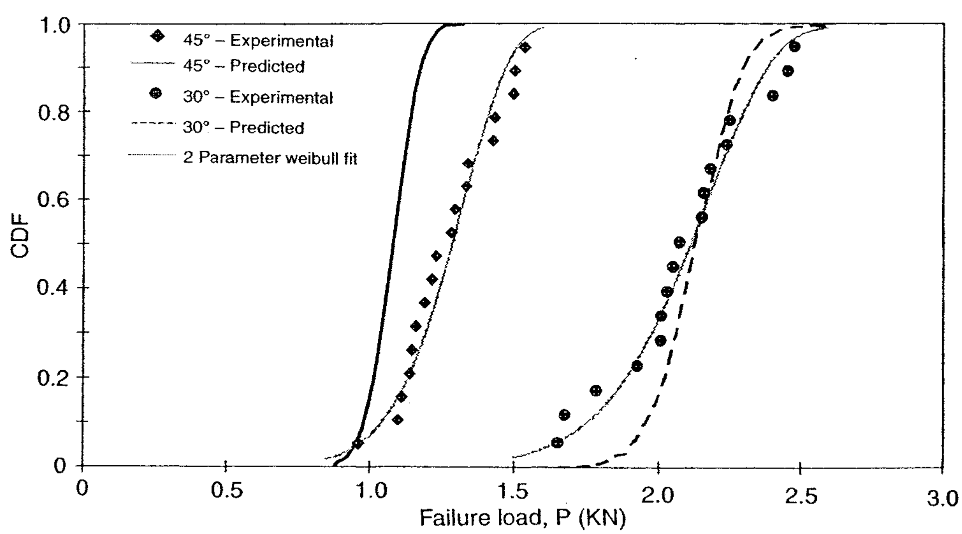

Fig. 4. CDF for off-axis bending failure load using method 1 (k from maximum likelihood approximation)

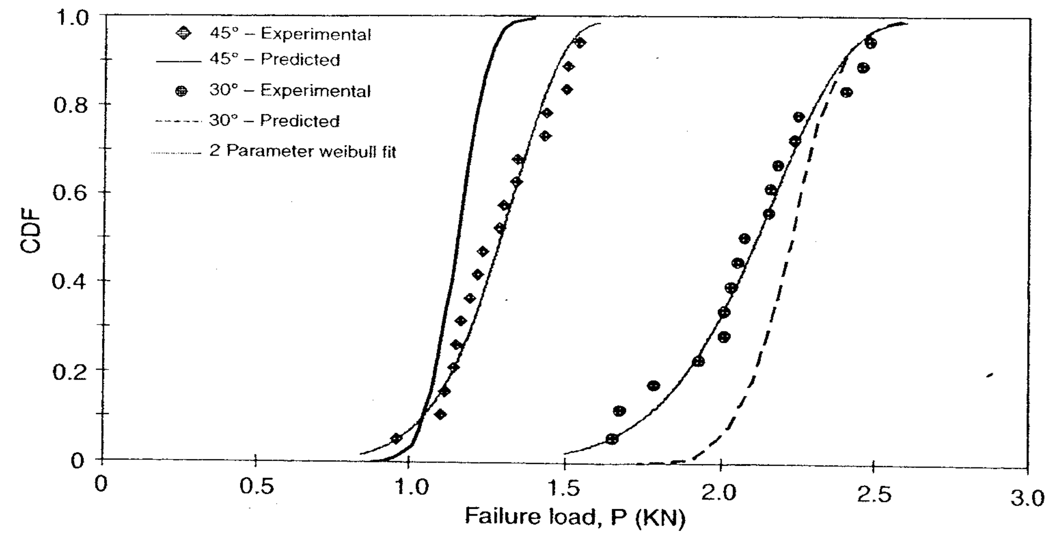

Fig. 5. CDF for off-axis bending failure load using method 1 (k from COV approximation)

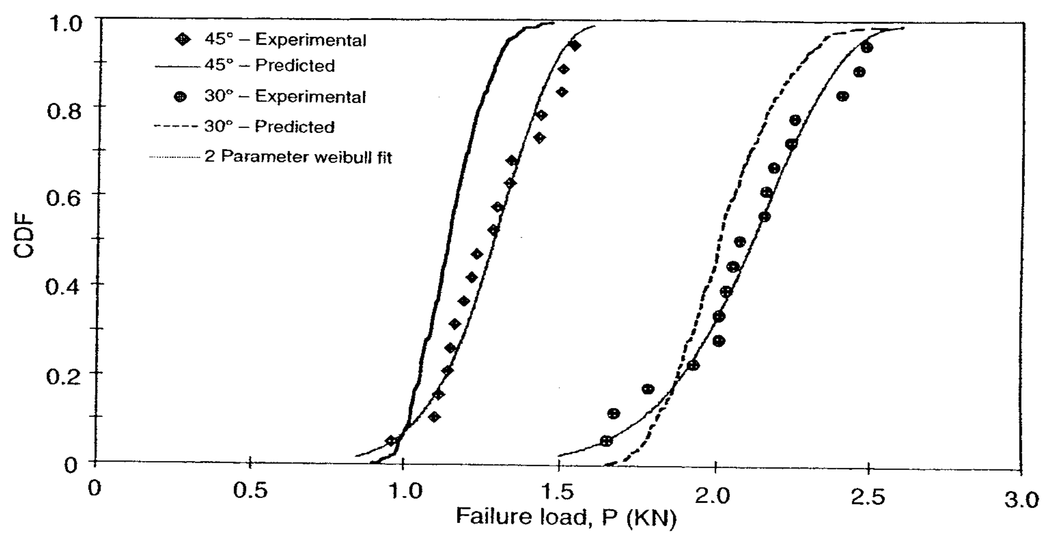

Fig. 6. CDF for off-axis bending failure load using method 2 (k from predicted tension strengths)

method 2 in comparison to method 1. Unlike method 1, method 2 incorporates the variability of *all* principal strengths by accounting for size effect using one shape parameter derived from a fit to the tension strengths predicted by Eq. (8). Method 1, on the other hand, accounts for size effect by using three separate shape parameters, each developed from relatively small samples sizes for the individual brittle principal strengths, X_t, Y_t, and S. Consequently, the accuracy of method 1 is highly dependent on the shape parameters for these strengths which, in turn, are closely linked to the respective strength's variability. As previously suggested, larger sample sizes would be very useful in improving the accuracy of the first method.

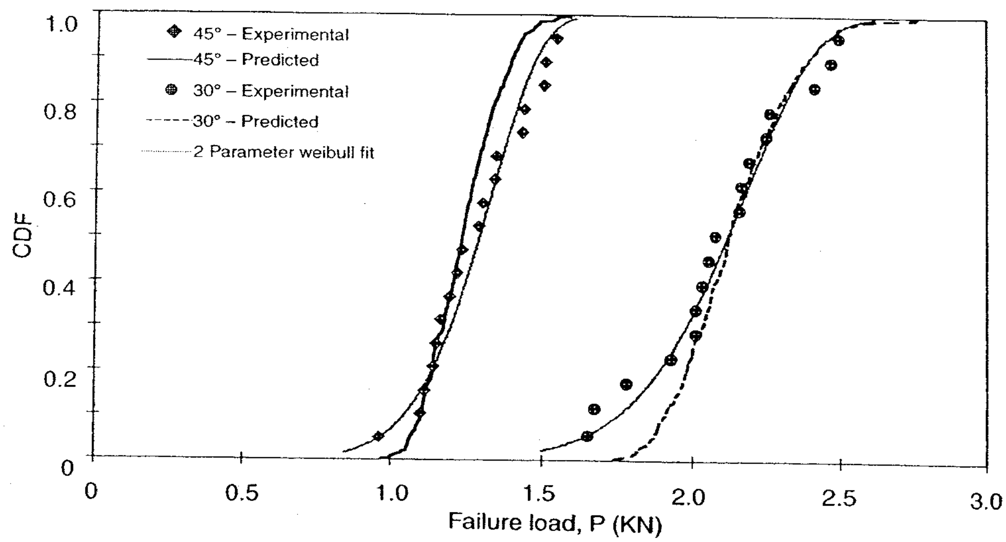

Fig. 7. CDF for off-axis bending failure load using method 2 (k from experimental bending tests)

To evaluate the sensitivity of method 2 to shape parameter used, another shape parameter, shown in Table 2 and obtained by fitting the experimental bending tests, was investigated. The results of this analysis are shown in Fig. (7). This figure illustrates very good agreement between method 2 and the experimental results with the best SSE of all analyses (SSE $= 0.08$). Since the shape parameter of the experimental bending tests is unavailable prior to the testing, this method cannot be considered as a prediction procedure. Rather, it illustrates that excellent agreement can be attained if sufficient information were available to estimate the variability of the strength properties.

Conclusions

This paper brings to light the fact that size effect must be considered when employing strength theories to predict the load carrying capacity of wood or wood composite members. Specifically, size effects must be accounted for when utilising experimentally obtained brittle strengths (representative of one material volume) in a strength criterion (representative of a different material volume). To demonstrate, the paper describes two distinct methods for coupling Weibull weakest-link theory with the Tsai-Wu strength theory. The analyses were performed assuming a linear elastic plane stress state under the general framework of probability theory. Strength parameters of the strength theory were described in terms of probability distributions and Monte Carlo simulations were employed to predict the load capacity of a center-point bending specimen using the material properties of Douglas-fir laminated veneer.

The first method described involved a straight forward transformation of brittle strengths from those representing the original test volume to those representing the small volume of the member to be analyzed with the strength criterion. The second approach entailed first using the strength criterion to predict ultimate load capacity of off-axis tensile specimens and then adjusting these predicted results using Weibull formulation.

Experimental tests were conducted to verify the proposed analytical techniques. Although results from both methods were reasonably accurate, the second method provided a more accurate estimation of ultimate off-axis bending strength. It was suggested that with larger experimental sample sizes for the brittle principal strengths of tension and shear, and hence a more accurate statistical representation of these strengths, the present accuracy of the prediction models could be improved.

Future studies endeavouring to predict the ultimate capacity of wood or wood composite members through strength theory modelling could benefit from the methods proposed herein. For example, the ideas put forth in this paper could ultimately be applied to other wood composite materials such as parallel strand lumber or complex wood composite assemblies such as wood I-joists or wood plate trusses to improve prediction accuracy.

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